Lagrangian Chaos and the Effect of Drag on the Enstrophy Cascade in Two-Dimensional Turbulence

Keeyeol Nam,* Edward Ott,*,† Thomas M. Antonsen, Jr.,*,† and Parvez N. Guzdar
Institute for Plasma Research, University of Maryland, College Park, Maryland 20742
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We investigate the effect of drag force on the enstrophy cascade of two-dimensional Navier-Stokes turbulence. We find a power law decrease of the energy wave number (k) spectrum that is faster than the classical (no-drag) prediction of k^{-3}. It is shown that the enstrophy cascade with drag can be analyzed by making use of a previous theory for finite lifetime passive scalars advected by a Lagrangian chaotic fluid flow. Using this we relate the power law exponent of the energy wave number spectrum to the distribution of finite time Lyapunov exponents and the drag coefficient.

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Two-dimensional Navier-Stokes fluid turbulence has been of interest for many years. Examples where two-dimensional (2D) Navier-Stokes turbulence is potentially relevant include soap film flow [1–6], rotating fluids [7], magnetically forced stratified fluids [8,9], plasma in the equatorial ionosphere [10], and the Earth's large scale (>500 km) atmospheric dynamics [11]. In these situations there are regimes where drag is an important physical effect [12]. In this case the describing Navier-Stokes momentum equation is

$$\frac{\partial \mathbf{v}}{\partial t} + \mathbf{v} \cdot \nabla \mathbf{v} = -\rho^{-1} \nabla p + \nu_1 \nabla^2 \mathbf{v} - \nu_0 \mathbf{v} + \mathbf{f}, \quad (1)$$

where ν_0 is the drag coefficient, ν_1 is the viscosity, $\mathbf{f}$ is an external forcing term, and incompressibility $(\nabla \cdot \mathbf{v} = 0)$ will be assumed. If the forcing is localized in k space with a characteristic wave number of k_f, it is well known that, in the absence of drag $(\nu_0 = 0)$, an energy cascade to larger scales, $E(k) \sim k^{-5/3}$ $(k < k_f)$, and an enstrophy cascade to smaller scales, $E(k) \sim k^{-3}$ $(k > k_f)$, are expected [13,14]. Our concern in this paper will be the effect of drag (the term $-\nu_0 \mathbf{v}$) on the enstrophy cascade in two-dimensional turbulence forced at low wave number (long wavelength). In the absence of drag $(\nu_0 = 0)$, many experimental and numerical results are consistent with the theoretical prediction of a k^{-3} dependence of the energy wave number spectrum $E(k)$ at wavelengths below the forcing wavelength [5,9,15] [while some other experimental and numerical results show that $E(k) \sim k^{-3}$ behavior is not conclusive yet [1–4,16]]. In the presence of drag, we find that there is still a power law dependence of the energy spectrum on k, but the power law exponent is greater than three,

$$E(k) \sim k^{-(3+\xi)}, \qquad \xi > 0.$$

This might be relevant to results of experiments where energy spectra steeper than k^{-3} were reported [1–4,16]. Furthermore, we find that the exponent increase ξ can be quantitatively related to the drag coefficient ν_0 and the chaotic straining induced by the flow. The key point in obtaining a theory for the exponent increase ξ is our argument (given subsequently) that the presence of drag fundamentally changes the character of the enstrophy cascade, making it possible to analyze the high wave number components by the use of a recent theory for the wave number spectrum of a finite lifetime passive scalar in a Lagrangian chaotic fluid flow. In particular, we will argue that the high k components of vorticity behave like a passive scalar advected by the Lagrangian chaotic low k components of the velocity field.

In our studies, we numerically solved the vorticity equation obtained by taking the curl of Eq. (1);

$$\frac{\partial \omega}{\partial t} + \mathbf{v} \cdot \nabla \omega = (-1)^{n+1} \nu_n \nabla^{2n} \omega - \nu_0 \omega + S_\omega(\mathbf{x}), \quad (2)$$

with $\mathbf{v} = \hat{z} \times \nabla \psi$, $\nabla^2 \psi = \omega$, where ψ is the stream function, $\omega = \hat{z} \cdot \nabla \times \mathbf{v}$ is the vorticity, and we have replaced the term $\nu_1 \nabla^2 \omega$ resulting from (1) by a hyperviscous damping, $(-1)^{n+1} \nu_n \nabla^{2n} \omega$. S_ω is a source function of the vorticity $(= \hat{z} \cdot \nabla \times \mathbf{f})$. Hyperviscosity ($n = 4$ for our numerical computations) is a commonly used device to enhance the wave number power law scaling range in numerical turbulence simulations. Our simulations employ a two-dimensional square domain, $[-\pi, \pi] \times [-\pi, \pi]$, with periodic boundary conditions, and $S_\omega(x, y) = \sin 2y$ is used for the source function of the vorticity. All the simulation results presented here use a spatial grid of 1024^2 and $\nu_4 = 5 \times 10^{-20}$.

The energy spectrum resulting from a simulation with $\nu_0 = 0.1$ is shown in Fig. 1. The energy spectrum is obtained by time averaging of instantaneous energy spectra measured at every 5 time units over a time duration of 40. The dashed line corresponds to $E(k) \sim k^{-3.5}$, where the exponent is from our theory which will be explained later. A clear power law behavior over almost one decade in k space is observable until the spectrum rolls off at high k's due to the hyperviscosity. Figure 2 shows plots of $k^3 E(k)$ versus k for three cases with $\nu_0 = 0, 0.1$, and 0.2 applied at high k $(k > 6)$, but with the same drag, $\nu_0 = 0.1$, applied at low k $(k \leq 6)$ in all three cases. The uppermost curve (circles) is for a high k drag of $\nu_0 = 0$, the

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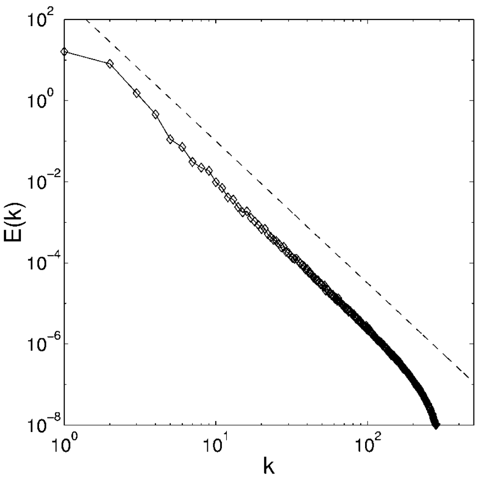

FIG. 1. Energy spectrum with a drag force coefficient $\nu_0 = 0.1$. The slope of the dashed line is -3.5 from the theory based on $G(h)$.

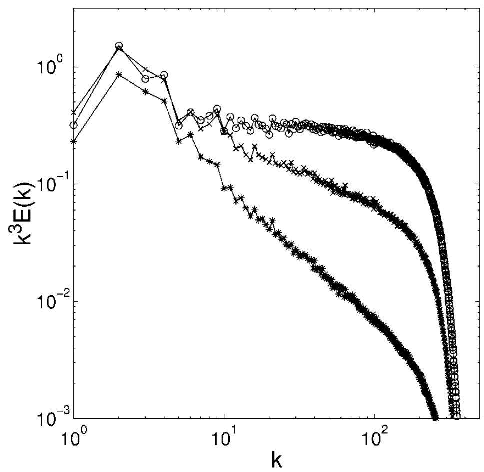

FIG. 2. Energy spectrum with various drag coefficients ν_0. The circles, the crosses, and the stars are for $\nu_0 = 0.0$, $\nu_0 = 0.1$, and $\nu_0 = 0.2$, respectively.

middle curve is for $\nu_0 = 0.1$, and the lowermost curve is for $\nu_0 = 0.2$. Applying the same low k drag allows us to compare effects of drag for high k's while keeping similar large scale dynamics of the flows [17]. This is important since we will show that, with drag, the major contribution to the straining of high k vorticity components is from the longest length scale (lowest k) flow components. While the energy spectrum for our $\nu_0 = 0.0$ case shows $\sim k^{-3}$ behavior, consistent with the classical Batchelor-Kraichnan prediction, the energy spectrum shows steeper power law behavior as we increase the drag coefficient. A summary of the simulation results appears in Table I.

In order to start to formulate a theory of $E(k)$ in the presence of drag, we first consider the straining rate, $\eta(r)$, on eddies of size r,

$$\eta(r) = \left\langle \frac{|\mathbf{v}(\mathbf{x} + \mathbf{r}) - \mathbf{v}(\mathbf{x})|^2}{r^2} \right\rangle^{1/2} \sim \sqrt{\int_{k_f}^{1/r} E(k) k^2 \, dk}\,, \tag{3}$$

where, in setting the lower limit of the integration over k at k_f, we have in mind the situation of our numerical simulation where k_f^{-1} is of the order of the system size and there is appropriate long wavelength damping [17]. In (3) the contribution of $E(k)$ from high wave numbers becomes negligible for large k (small r) if $E(k) \sim k^{-(3+\xi)}$ with $\xi > 0$ (which is the case when drag is present). For the classical case without a drag term where $\xi = 0$, the above quantity diverges logarithmically at large k (small r) and this leads to Kraichnan's prediction of a weak logarithmic correction to the power law energy spectrum, $E(k) \sim k^{-3} \log^{-1/3}(kL)$ [18]. In contrast, if $E(k) \sim k^{-(3+\xi)}$ with $\xi > 0$, then for small r the straining rate becomes constant, $\int_0^{1/r} E(k)k^2\, dk \simeq \int_0^{\infty} E(k)k^2\, dk$, indicating that the straining of smaller scales comes from the larger scales. Thus, for the $\xi > 0$ case, it is a good approximation to assume that the vorticity components at sufficiently high k are passively advected by the chaotic velocity field of large structures and this approximation becomes better and better as k is increased. In treating the case with drag we shall use Lyapunov exponents. We thus need to discuss the relevance of Lyapunov exponents to our problem. Crudely, we estimate the Lyapunov exponent h as the straining rate at vanishingly small separation, i.e., $r = 0$ in Eq. (3), $h \sim [\int_{k_f}^{\infty} k^2 E(k)\, dk]^{1/2}$. In the presence of viscosity (or hyperviscosity), the resulting high k cutoff to $E(k)$ implies a finite Lyapunov exponent. However, depending on the exponent in the power law dependence of $E(k)$, different wave numbers, i.e., different scale lengths will determine the value of the Lyapunov exponent. With no drag we expect $E(k) \sim k^{-3}$ and the Lyapunov exponent is dependent on contributions from flow components with wave numbers down to the viscous cutoff scale. [This is signaled by the logarithmic dependence of $\int_{k_f}^{1/r} E(k)k^2\, dk$ on r.] This means that the Lyapunov exponents only describe straining of scales smaller than the viscous cutoff scale, and hence are not relevant to the enstrophy cascade. On the other hand, if $E(k) \sim k^{-(3+\xi)}$ with $\xi > 0$, as we claim for the case with drag, then the situation is different. When $\xi > 0$, Lyapunov exponents are relevant in the absence of a high k cutoff and are determined by the largest scale flow components. Thus, for $\xi > 0$, the Lyapunov exponents describe the straining of scales small compared to k_f^{-1}, which, however, may still be large compared to a viscous cutoff.

TABLE I. Comparison of the power law exponents from the direct numerical simulations, $(3 + \xi_{NS})$, to the theoretical results, $(3 + \xi_{\rm th})$. The spectral ranges linear fitted to measure the power law exponents are $\log_{10} k = [1.0, 2.0]$ for the direct numerical simulation.

$\nu_0(k > 6)$	$3 + \xi_{NS}$	$3 + \xi_{\rm th}$
0.0	3.1	3.0
0.1	3.6	3.5
0.2	4.1	4.0

The discussion in the previous paragraph implies that the situation for ω at high k is the same as the situation for a passive scalar ϕ convected by a Lagrangian chaotic flow $\mathbf{v}$ in the presence of a loss term such that ϕ obeys an equation [analogous to (2)] of the form $\partial\phi/\partial t + \mathbf{v}\cdot\nabla\phi = D\nabla^2\phi - \nu_0\phi + S_\phi(\mathbf{x})$, where $-\nu_0\phi$ is the loss term. (This type of spectral passive scalar problem is of interest for certain reacting chemical scalars and also for plankton concentration in the ocean [19–21].) The wave number spectrum for such a passive scalar has recently been considered in Ref. [20], and we can make direct use of that reference. According to Ref. [20], the wave number power spectrum of ϕ behaves as $E_\phi(k) \sim k^{-(1+\xi)}$, where ξ is determined by the probability distribution function $P(h,t)$ for finite time (t) Lyapunov exponents (h). In order to define this function, consider a disk of fluid of infinitesimal radius δr_0 originally located at a point $\mathbf{r}$, and evolve the disk forward for a time t. At time t the disk has been deformed into a differential ellipse whose major radius is δr_t. The time t finite time Lyapunov exponent for the initial point $\mathbf{r}$ is $h(\mathbf{r},t) = \ln(\delta r_t/\delta r_0)$. $P(h,t)\,dh$ is the probability that $h(\mathbf{r},t)$ is between h and $h + dh$ given that the initial point $\mathbf{r}$ is chosen randomly. For large t, it can be shown that $\ln P(h,t)$ has an asymptotic form $\ln P(h,t) \sim -tG(h)$, where $G(h)$ is concave up $[G''(h) > 0]$ and has a minimum at $\bar{h}$ [i.e., $G'(\bar{h}) = 0$] with $G(\bar{h}) = 0$, where $\bar{h}$ is the usual infinite time Lyapunov exponent [22]. According to [20], the spectral exponent is determined from $G(h)$,

$$\xi = \min_h\{[G(h)/h] + (2\nu_0/h)\}. \tag{4}$$

As argued above, at $k \gg k_f$ we can treat ω as a passive scalar in a Lagrangian chaotic fluid flow so that $E_\omega \sim k^{-(1+\xi)}$. Also, since $\nabla\cdot\mathbf{v} = 0$, $\omega = \hat{z}\cdot\nabla\times\mathbf{v}$, we have $E(k) = k^{-2}E_\omega(k)$, and $E(k) \sim k^{-(3+\xi)}$ with ξ given by (4).

We test the applicability of this theory to the enstrophy cascade with drag by numerically obtaining $G(h)$ from a histogram approximation to the finite-time Lyapunov exponent distribution $P(h,t)$. The latter is obtained by computing h for 4×10^4 initial conditions spread uniformly in space. A cubic polynomial fit to $t^{-1}\ln P(h,t)$ is utilized to obtain $G(h)$. Theoretical power law exponents obtained from Eq. (4) and the numerical $G(h)$ are compared with the results from full numerical simulations in Table I. Within the accuracy of our computations reasonable agreement is obtained.

It is also of interest to consider the possible power law scaling of the vorticity structure functions,

$$\langle|\omega(\mathbf{x}+\mathbf{r}) - \omega(\mathbf{x})|^q\rangle \sim r^{\zeta_q}, \tag{5}$$

for small $r = |\mathbf{r}|$ and $r >$ (viscous cutoff length). Recently the structure functions for Lagrangian chaotic advection of finite lifetime passive scalars have been considered [23]. By our previous arguments we can directly apply the result of Ref. [23] to the vorticity structure functions for two-dimensional turbulence with drag,

$$\zeta_q = \min_h\{q, [G(h)/h] + (q\nu_0/h)\}, \tag{6}$$

which generalizes (4). That is, ζ_2 agrees with ξ in (4).

For the case without drag ($\nu_0 = 0$), Eq. (6) indicates that $\xi_q = 0$ for all q [recall that $G(\bar{h}) = 0$]. This indicates the absence of small scale intermittency in the enstrophy cascade for $\nu_0 = 0$, in agreement with previous work (see [9] and references therein). This situation is in striking contrast with three-dimensional turbulence where intermittency is clearly present. Equation (6), however, shows the important result that drag ($\nu_0 > 0$) leads to intermittency also in the two-dimensional case ($\zeta_q \neq 0$ and not proportional to q).

In conclusion, we have considered the effect of linear drag on the enstrophy cascade of two-dimensional Navier-Stokes turbulence. We find that, as in the case without drag, enstrophy cascade results in power law behavior of the energy spectrum. However, the power law exponent is different from the classical Batchelor-Kraichnan prediction (-3) and is determined by the distribution of finite time Lyapunov exponents and the drag coefficient.

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*Department of Physics.
†Department of Electrical Engineering.

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